Acoustic Instabilities in a Constant Flux Gas Core Nuclear Rocket

HARRY MCNEIL* AND MARTIN BECKER†
Rensselaer Polytechnic Institute, Troy, N. Y.

The gaseous core nuclear rocket has a potential for significant improvement in propulsion capability over the solid fuel nuclear rocket because the gaseous system may permit operation at much higher temperatures. The similarity of the core of the gaseous core nuclear rocket to the combustion chamber of a chemical rocket invites speculation that acoustic instabilities, which have been troublesome in chemical rocket systems, may also occur in the gaseous core nuclear rocket system. Such instabilities can occur in compressible fluids with pressure dependent heat generation. A simplified and idealized physical model of a gaseous core nuclear rocket is presented in which the gasdynamic field is perturbed by wave phenomena while the neutron field is assumed constant. A simple stability criterion is obtained from this model in terms of the equilibrium values of the state variables and the size of the system. The criterion is aplied to a reference case with the result that instability is predicted for nuclear rockets well within the size envelope presently proposed in conceptual designs.

Nomenclature

L = differential operator
∇ = gradient operator
Ψ_j = function
$\dfrac{D}{Dt}$ = convective derivative
s = complex frequency
k = wave number
σ = order of magnitude
ρ = density
x = spatial coordinate
μ = velocity
Ψ_{ij} = stress tensor
e_m = energy per unit mass (internal plus kinetic)
R = gas constant
T = temperature
p = pressure
γ = ratio of specific heats
Q = nuclear heat source
q = radiation vector
δ = variation operator
u = speed
a = speed of sound
ϵ = fission power constant
Σ_f = fission cross section
ϕ = neutron flux
Γ = dimensionless gain parameter
$D(k,s)$ = dispersion function
λ = wave length
λ^* = normalized wave length
a = sound speed
K = Rosseland opacity
ω = angular frequency

Subscripts and superscripts

$'$ = perturbed value
0 = equilibrium value
p = pth component
c = stability boundary value

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* Graduate Student, Division of Nuclear Engineering and Science; now Assistant Project Engineer, Pratt and Whitney Aircraft Division of United Aircraft Company.

† Associate Professor, Division of Nuclear Engineering and Science.

Introduction

THE mission requirements of interplanetary space travel have led propulsion technology to the chemical rocket system useful in earth orbit and lunar exploration. However, the chemical rocket system may not be adequate for some advanced missions. Therefore, more effective propulsion devices are being developed.[1] The solid core nuclear rocket looks promising, but even as the development and testing of this rocket continues, serious consideration is being given to advanced nuclear systems which can extend performance by operating at temperatures far above those of the solid core rocket. The gaseous core nuclear rocket, in which fuel is held in gaseous form, improves performance by extending the maximum propellant exhaust temperature beyond the limits imposed by the melting point of the solid fuel element in the Rover type solid core nuclear systems.[2,3]

In some respects, the gaseous core is similar conceptually to the combustion chamber of a chemical rocket. Because of the similarity, one may ask whether there could be dynamic problems in the nuclear system analogous to the combustion instabilities that are of concern in chemical systems. Putnam[4] describes the disastrous effect of combustion instabilities due to acoustic waves in chemical systems, but in fact any gaseous system which contains an intrinsic pressure-dependent heat source may be subject to acoustic instabilities. The chemical systems technology, therefore, provides a starting point for the analysis of a gas core cavity reactor. However, although there are similarities between a combustion chamber and a gaseous core, the differences between them also are significant. In the chemical system, heat is removed by convection when the combustion products leave the combustion chamber. In the nuclear system, heat is transferred by radiation to the propellant. In addition, chemical kinetics and their associated feedback mechanisms differ from neutron dynamics and the feedback mechanisms of the gas core nuclear system.

The possibility of acoustic instabilities in nuclear reactors was recognized by Welton[5] as a possible hazard in the case of homogeneous reactor systems. He obtained an integral formulation of stability criteria for a homogeneous nonreflected thermal system. The dynamics of this one-region system with internal moderation, however, is quite different from the gas core cavity reactor system that we are investigating.

Most work to date on gaseous rockets has been concerned with concept feasibility. Recently, some work has appeared relevant to the dynamics of the cavity reactor system. Smith[6] has investigated the generation of a pressure wave by neutron irradiation. Podney and Smith[7] have analyzed the prompt neutron kinetics of a spherical cavity reactor but included only neutronics (no feedback) in a simplified integral equation formulation in which cavity density changed uniformly with time.

Our review of the literature indicated that the potentially important effect of acoustic wave propagation in the gaseous core of the nuclear rocket has not as yet been investigated. The probable reason for this is preoccupation with problems of fuel retention and separation which bear directly upon the feasibility of gaseous reactor development. The purpose of this investigation is to determine if acoustic instabilities can occur in a gaseous core rocket and to determine approximate stability criteria in terms of the equilibrium values of the system parameters. A simplified and idealized model is presented which emphasizes the essential characteristics of the gaseous core rocket. This model describes the interaction of the gasdynamic, radiation, and neutron fields for the special case where the neutron field is constant. An extension of the model presents, in terms of a nondimensional gain parameter, a simplified description of the effect of neutron field perturbation on stability. The dynamics of a gaseous core rocket in which neutronics is considered in detail has been investigated[8] and will be presented in a subsequent paper. In addition, the gas is assumed to be composed of electrically neutral particles. Plasma effects are discussed elsewhere.[8]

Field Stability Criteria

Complex dynamic systems can be described by a set of field variables which can express the essential characteristics of the system in terms of the interaction of representative fields. At equilibrium or steady state, the field variables can be considered to be coordinates in a state space. When one or more of the variables is perturbed, the set may be stable and return to the equilibrium state, or the variables may be unstable and diverge with the passage of time or the traverse in space. The essential characteristics of the fields can be expressed as a set of nonlinear partial differential equations derived from the conservation laws of mass, momentum, energy, neutronics, and radiation. In terms of an operator with arguments consisting of gradients and time derivatives, the field equations can be written as

$$\sum_{j=1}^{n} L_{ij}\left(\Delta,\frac{\partial}{\partial t}\right)\Psi_j(r,t) = 0 \qquad (1)$$

The functions Ψ_j are the field variables and depend on position and time. Small signal perturbation theory can be used to investigate stability. The functions can be expanded in a small parameter δ about the equilibrium point:

$$\Psi_j(r,t) = \Psi_j(r,0)\delta^0 + \Psi_j'(r,t)\delta' + \ldots \qquad (2)$$

This expansion is substituted into the field equations (1) and, by equating the coefficients of like powers of δ and ignoring terms of $\sigma(\delta^2)$ and above, an equilibrium and perturbed set of linear equations can be obtained as follows:

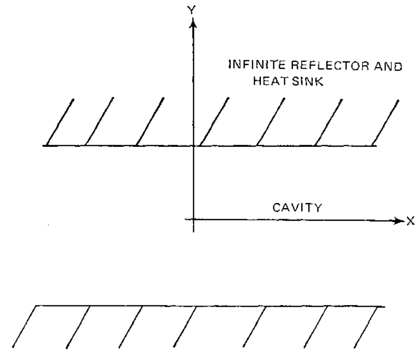
Equilibrium

$$\sum_{=1}^{n} N_{ij}(\nabla)\Psi_j(r,0) = 0$$

Perturbed: (3)

$$\sum_{j=1}^{n} M_{ij}\left(\Delta,\frac{\partial}{\partial t}\right)\Psi_j'(r,t) = 0$$

Fig. 1 Cavity geometry and coordinate system.



The perturbed set is Fourier transformed in space and Laplace transformed in time, resulting in a linear set of algebraic equations in terms of the transformed field variables. A solution is obtained by initiating a disturbance in at least one function and solving the set of algebraic equations by Cramer's rule. The determinant of the algebraic equations is defined as the dispersion function, and the zeros of the dispersion function are the roots of the dispersion equation. The roots correspond to the complex frequencies of time-dependent exponential functions and in general depend on both the equilibrium parameters and the wave number k (which can be related to the size of the system). The system will be stable when the real parts of these complex frequencies are negative, marginally stable when they are zero, and unstable when they are positive. Hence the stability criterion can be expressed in terms of the complex frequency as

$$Re[s_i] \leq 0 \qquad i = 1,2,\ldots n \qquad (4)$$

The equality is satisfied for the critical wave number and yields a point on the stability boundary, a point of marginal stability. The stability boundary is, therefore, on the locus of pure imaginary roots of the dispersion equation for critical wave numbers.

Conservation Laws and the Dispersion Equation

The conservation laws of mass, momentum, energy, and state can be expressed in tensor notation with the summation over the tensor indices $i = 1,2,3$; $j = 1,2,3$

$$\begin{aligned}
\partial\rho/\partial t + (\partial/\partial x_i)(\rho u_i) &= 0\\
\rho(D/Dt)u_i - (\partial/\partial x_j)\Psi &= 0\\
\rho(D/Dt)e_m - (\partial/\partial x_j)(u_i\Psi_{ij}) - Q - (\partial/\partial x_i)q_i &= 0 \qquad (5)\\
p - \rho RT &= 0\\
Q - \epsilon\Sigma_f\phi &= 0
\end{aligned}$$

The aforementioned conservation laws are applied to the cavity reactor model shown in Fig. 1. The system consists of an infinite cavity bounded in the transverse direction by a neutron reflector and infinite heat sink, fueled with U^{235} gas and critical at power level Q_0. The cavity gas is stationary and initially uniform. No gradients exist except for transverse gradients consistent with the heat removal requirements at steady state. The specialized case of plane wave propagation will be investigated by assuming that transverse dependence can be separated from longitudinal dependence so as to yield perturbation equations involving only time and the X direction. The reduction to a longitudinal problem is described in the Appendix. Any effects of mode coupling between longitudinal and transverse waves thus are not considered. The damping effects of heat conduction and viscosity are small (as will be discussed later) so these phenomena will be neglected.

The linearized equations are obtained by expansion of the state variables as previously described to obtain the following

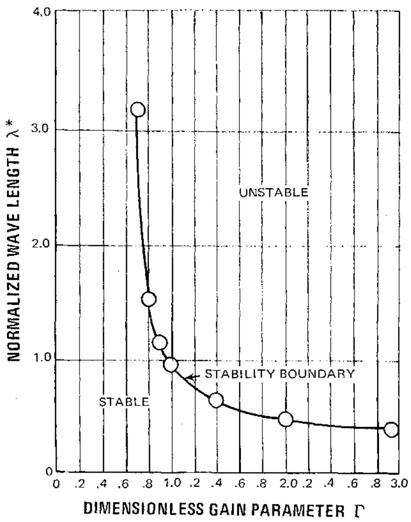


Fig. 2 Effect of variations in the gain parameter on the stability boundary.

perturbed set:

$$\partial p'/\partial t + \rho_0(\partial u'/\partial x) = 0, \; \rho_0(\partial u'/\partial t) + \partial p'/\partial x = 0$$

$$\rho_0 C_p(\partial T'/\partial t) - \partial p'/\partial t = Q' + K_0(\partial^2 T'/\partial x^2) - Q_0 T'/T_0 \tag{6}$$

$$p'/p_0 = \rho'/\rho_2 + T'/T_0$$

$$Q' = \epsilon \Sigma_{f0}\phi_0(\Sigma_f'/\Sigma_{f0} + \phi'/\phi_0)$$

In the expression for the energy equation, the first term on the right is the nuclear heat source, and its linearized expansion is written as the last equation in the preceding set. The second term refers to heat radiation in the longitudinal direction and the last term indicates heat transfer in the transverse direction. The central problem in estimating the heat transfer in the gaseous core of the rocket is in estimating the opacity of the cavity. In a gas as dense as a critical concentration of U^{235}, the photon mean free path is extremely small and consequently the photon transport equation simplifies to the optically thick or diffusion theory approximation. Estimates of the Rosseland mean opacity for defining an effective thermal conductivity have been obtained by Krascella.[9]

The expression for the nuclear heat source shows the dependence of fission-induced heat generation on perturbations in the fission cross section and neutron flux in the cavity. In this investigation, we will consider only the effect of power fluctuations due to density dependent perturbations in the fission cross section. For convenience, the effect of density perturbations will be presented in terms of a dimensionless gain parameter Γ;

$$Q' = Q_0\Gamma\rho'/\rho_c \qquad \Gamma \geq 0 \tag{7}$$

The parameter Γ can be adjusted to correspond to physically realizable situations. For example, $\Gamma = 0$ corresponds to zero power perturbation and, therefore, no source of instability. When $\Gamma = 1$, we have the case of no neutron field perturbation (constant flux). A simplified qualitative account of the influence of the neutron field is obtained by examining the dependence of stability criteria on Γ (though a complete study of neutronic feedback caused by perturbations in the neutron field would necessarily result in changes in both modulus and argument of the power transfer function). These results will be discussed later.

The dispersion equation is obtained by transforming the field variables and performing other operations as previously described to obtain

$$D(k,s) = \sum_{i=1}^{4} A_i(k)s^{i-1} = 0 \tag{8}$$

where

$$A_4 = 1$$
$$A_3 = K_0T_0(\gamma - 1)k^2/\rho_0 + Q_0(\gamma - 1)/p_c$$
$$A_2 = p_0\gamma k^2/\rho_0$$
$$A_1 = K_0T_0(\gamma - 1)k^4/\rho_0 + Q_0(\gamma - 1)k^2(1 + \Gamma)/\rho_0$$

When the dispersion equation is a real polynomial, the existence of roots in the positive half-plane can be determined by using Routh's criterion. One obtains the following stability criterion:

$$k^2 \geq Q_0(1 + \Gamma - \gamma)/K_0T_0(\gamma - 1) \tag{9}$$

In terms of wave length; when $\lambda = 2\pi/k$

$$\lambda^2 \leq 4\pi^2K_0T_0(\gamma - 1)/Q_0(1 + \Gamma - \gamma) \tag{10}$$

We define the normalized wave lengths as $\lambda^* = \lambda(\Gamma)/\lambda(1)$ and plot Eq. (10) in Fig. 2 using this parameter. When Γ is reduced below one, there is a rapid trend toward stability.

One also may consider the case of a complex wave length and real frequency. The dispersion equation is transformed by making the substitution $s = i\omega$ and solving the resulting bi-quadratic equation for the complex wave length. A closed form solution is easily obtained by making the assumption that heat transfer and the heat source are $\sigma(\delta)$ and that higher order terms can be neglected. When $\Gamma = 1$ the roots are

$$k_{1,2} = \pm\frac{\omega}{a_0}\left[\frac{q_0{}^4\rho_0}{2K_0T_0(\gamma - 1)\omega}\right]^{1/2}(1 - i)$$

$$k_{3,4} = \pm\frac{\omega}{a_0}\left\{1 - i\left[\frac{K_0T_0(\gamma - 1)^2\omega}{2\rho_0 a_0{}^4} - \frac{(2 - \gamma)(\gamma - 1)Q_0}{2\rho_0 a_0{}^2\omega}\right]\right\} \tag{11}$$

The first pair of roots have an associated wave which might be called a heat wave.[10] It is clearly a sound wave which is modified by frequency, and, therefore, is dispersive. It depends on heat transfer through the heat-transfer coefficient. When the heat transfer increases, the roots tend to vanish and the waves tend to become undamped, and nonpropagating. As the heat transfer decreases the roots become larger, resulting in increased damping and eventual disappearance of the wave. This wave does not contribute to instability.

The second pair of roots represents a sound wave which has a phase velocity equal to the equilibrium speed of sound. The attenuation and amplification is described by the imaginary part of the complex wave number in which the first term represents spatial attenuation due to heat transfer, whereas the second term indicates spatial amplification caused by the density-dependent heat source. At low frequencies, the heat source predominates and the system is unstable, but as the frequency increases, the attenuation grows and drives the system stable. The cross over point is obtained by setting the imaginary components of the roots to zero. The resulting wave is undamped and, therefore, we can substitute $\omega_c = a_0k_c$ at the stability boundary. When these substitutions have been made, and the equations solved for the critical wave number, we obtain again the stability criterion shown in Eq. (10) for the special case when $\Gamma = 1$.

We have obtained a stability criterion for an acoustic wave in an infinitely long cavity. We must relate the critical wavelength for instability to the critical size for instability of a finite-length reactor. In a system of finite size, we are interested in the growth rate of the most nearly unstable standing wave that can exist. The relationship between wavelength and size depends on the boundary conditions appropriate for the finite cavity. The nature of these boundary conditions is determined by the presence and specific characteristics of nozzles, injection systems, etc. The formulation of these boundary conditions has been discussed for chemical systems.[11] Since in this study, we do not wish to specialize to specific gaseous reactor concepts or designs, we shall assume for simplicity that the fundamental standing wave will have a wave length twice the length of the cavity (i.e., the size of the cavity is the distance between nodes of the wave).

To determine the practical implications of the stability criterion, a system with typical characteristics (given in Table 1)[12] was analyzed. The critical wavelength was found to be about 100 cm (critical frequency of 9.1×10^3 rad/sec), so we would expect instability in a system more than 50 cm long. However, a typical value of core length is 300 cm. Thus, according to our stability criterion, a typical reference design would be unstable with respect to acoustic waves. The results were found not to be unduly sensitive to the specific numbers associated with the reference case.

Effect of Heat Conduction and Viscosity

In formulating the conservation equations, heat conduction and viscosity were neglected. It is reasonable to neglect heat conduction because the thermal conductivity is roughly three orders of magnitude smaller than the effective conductivity for radiation heat transfer. The damping effect of viscosity on an acoustic wave is comparable to the damping effect of thermal conductivity[13] and is therefore small compared with the effect of radiation heat transfer. Thus, for the purpose of the analysis of acoustic instability, one should be able to neglect both heat conduction and viscosity.

Summary and Conclusions

In this paper, a simple and idealized model of a gaseous core nuclear rocket has been presented for the purpose of studying acoustic instabilities. On the basis of this model, it was found that typical reference designs would be unstable. Although the model used was indeed simple, we believe that conclusions may be made on the basis of the order of magnitude of the results. Therefore, we conclude that acoustic instability is a potential problem for gaseous reactors that merits further and more detailed study.

Appendix: Reduction to a Longitudinal Problem

In order to proceed from Eq. (5) to the longitudinal wave equation of Eq. (6), it is necessary to eliminate the transverse temperature distribution. Let us illustrate the procedure used with reference to a cavity with one transverse dimension. Linearizing Eq. (5) without averaging would yield

$$\rho_0 C_v \frac{\partial T'}{\partial t} = -p_0 \frac{\partial u'}{\partial x} + Q' + K_0 \frac{\partial^2 T'}{\partial x^2} + K_0 \frac{\partial^2 T'}{\partial y^2} \quad \text{(A1)}$$

We wish to operate on Eq. (A1) in such a way as to obtain Eq. (6), repeated here for convenience as

$$\rho_0 C_p(\partial T'/\partial t) - \partial p'/\partial t = Q' + K_0 (\partial^2 T'/\partial x^2) - Q_0 T'/T_0 \quad \text{(A2)}$$

Taking the difference between Eqs. (A1) and (A2) yields

$$\rho_0 R \partial T'/\partial t - \partial p'/\partial t - p_0 \partial u'/\partial x = -K_0 \partial^2 T/\partial y^2 - Q_0 T'/T_0 \quad \text{(A3)}$$

It is a simple matter to show that the left-hand side of Eq. (A3) vanishes. Note that, from continuity and state equations, it follows that

$$\rho_0 \frac{\partial u'}{\partial x} = -\frac{p_0}{\rho_0}\frac{\partial \rho'}{\partial t} = -p_0\left[\frac{1}{p_0}\frac{\partial p'}{\partial t} - \frac{1}{T_0}\frac{\partial T'}{\partial t}\right] \quad \text{(A4)}$$

Equation (A3) thus reduces to

$$K_0(\partial^2 T'/\partial y^2) + Q_0 T'/T_0 = 0 \quad \text{(A5)}$$

Table 1 Equilibrium parameters for reference case

Symbol	Name	Value
K	Rosseland opacity (erg/°K-cm-sec)	0.380×10^7
P	Pressure (dynes/cm²)	0.505×10^8
ρ	Density (g/cm³)	0.400×10^{-2}
T	Temperature (°K)	0.1357×10^5
γ	Ratio of specific heats	0.167×10^1
Q	Fission power (erg/sec-cm³)	0.44×10^9

To determine the validity of Eq. (A5), consider first the steady-state (unperturbed case) equation

$$K_0 \partial^2 T_0/\partial y^2 + Q_0 = 0 \quad \text{(A6)}$$

(Recall that initial properties are uniform, so that x dependence is constant). Let us express $T_0(x,y)$ in the form

$$T_0(x,y) = T_0 f(y) \quad \text{(A7)}$$

normalized so that

$$\int dy f(y)/\int dy = 1 \quad \text{(A8)}$$

Let us assume that, analogous to Eq. (A7),

$$T'(x,y,t) = T'(x,t)f(y) \quad \text{(A9)}$$

Substituting Eq. (A.9) into Eq. (A.5) yields

$$K_0 T'(x,t)\partial^2 f/\partial y^2 + (Q_0/T_0)T'(x,t)f(y) = 0 \quad \text{(A10)}$$

After cancelling $T'(x,t)$, we are left with the differential equation that defines $f(y)$ and Eq. (A10) is satisfied identically.

In effect, the procedure followed is to assume that heat transfer to the surroundings in the transverse direction can be represented by an effective heat-transfer coefficient which is constant for small perturbations about the reference state for which the coefficient is defined.

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